7. Problem sheet for Set Theory, Winter 2012

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Problem 23. (Cantor normal form) Show that for every ordinal α, there are a unique $k \in \omega$ and unique tuples $(m_0, ..., m_k)$ of natural numbers and $(\alpha_0, ..., \alpha_k)$ of ordinals with $\alpha = \omega^{\alpha_0} m_0 + \omega^{\alpha_1} m_1 + ... + \omega^{\alpha_k} m_k$ and $\alpha_0 > ... > \alpha_k$.

Problem 24. (Transversals) Prove that AC is equivalent to the following statement: For every set X and every equivalence relation E on X there is a transversal for E, i.e. a subset of X which intersects every equivalence class in exactly one point.

Problem 25. (Cardinality) Prove:

(a) $\mathbb{R} \sim [0,1] \sim (0,1)$.
(b) $\mathbb{R} \sim {}^{\omega}2$.
(c) ${}^{\omega}[0,1] \sim [0,1]$.

Problem 26. (Cantor-Schröder-Bernstein theorem) Suppose $B \subseteq A$ and $f\colon A \to B$ is injective. We define $A_0 = A$, $A_{n+1} = f[A_n]$, $B_0 = B$, $B_{n+1} = f[B_n]$ for $n \in \omega$ by recursion. Let

$$g(x) = \begin{cases} f(x) & \text{if } x \in A_n \setminus B_n \text{ for some } n, \\ x & \text{otherwise.} \end{cases}$$

Show that $g\colon A \to B$ is bijective.

There are 6 points for each problem. Please hand in your solutions on Monday, November 26 before the lecture.